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Thermodynamic Bethe ansatz for planar AdS/CFT: a proposal

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Abstract
Moving from the mirror theory Bethe–Yang equations proposed by Arutyunov and Frolov, we derive the thermodynamic Bethe ansatz equations which should control the spectrum of the planar AdS_5/CFT_4 correspondence. The associated set of universal functional relations (Y-system) satisfied by the exponentials of the TBA pseudoenergies is deduced, confirming the structure inferred by Gromov, Kazakov and Vieira.

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1. A bird's-eye view between integrability and AdS/CFT

A very peculiar phenomenon in modern theoretical physics has been taking place at the encounter of two branches: on one side the subject of quantum/statistical two-dimensional integrability [1] and on the other the gauge/string correspondences [2] in their planar case. Actually, the entrance of integrability into the realm of reggeised gluons of infinite colour QCD in its leading logarithmic approximation was already observed by Lipatov in [3].

More specifically, the AdS/CFT conjecture relates, by a strong/weak coupling duality, a type IIB superstring theory on the curved spacetime $AdS_5 \times S^5$ and the conformal $\mathcal{N} = 4$ super-Yang–Mills (SYM) theory in four dimensions on the boundary of AdS_5 [2]. As a consequence and particular case, the energy of a specific string state ought to be equal to the anomalous dimension of the corresponding local gauge invariant operator in the quantum field theory. Yet, the mechanism of integrability in this triadic relation is not fully understood. For sure, the discovery of integrability in the classical string theory was a great achievement [4], both from the conceptual and the practical (i.e. calculative) points of view.

On the SYM theory side of the correspondence, the large colour number limit $N \to \infty$ is taken keeping the 't Hooft coupling $Ng^2_{\mathrm{YM}} = \lambda = 4\pi^2 g^2$ finite, with g proportional to free string tension. In this limit only the planar Feynman diagrams and single trace composite operators survive. Besides the pioneering interpretation of [5] in terms of an $sl(2)$ spin chain (in the QCD case), the constituent operators in the purely scalar sector at one loop have been unveiled to correspond to the degrees of freedom of an integrable $so(6)$ spin chain, thus making the mixing matrix (or dilatation operator) to coincide with this integrable $so(6)$ spin Hamiltonian [6]. Being integrable, the spectrum of this Hamiltonian comes out by means of the Bethe ansatz (BA) (in one of its various forms) [1] and described by the so-called Bethe ansatz equations for the 'rapidities' which parametrize the operators in the trace. Albeit a description of the dilatation operator at all loops as a spin chain Hamiltonian is still missing, the integrability has been showing up in the form of spin-chain-like Bethe equations (for g dependent rapidities still parametrizing the operators in the trace, likewise to the one-loop case), which are valid at least in the *asymptotic regime* of large quantum numbers (cf below). Eventually, a set of equations for the whole theory has been proposed by Beisert and Staudacher [7]. Computationally, the BA energy, $E(g)$, yields the anomalous part of the conformal dimension:

$$\Delta = \Delta_{\text{bare}} + g^2 E(g), \tag{1.1}$$

where Δ_{bare} is the bare or classical dimension. As said before, this quantity must also be given by the quantum energy of a suitable string state ($E_{\text{string}} = \Delta$). By a semiclassical procedure on the string sigma model, this fact has opened a road to fix a phase factor, the so-called dressing factor, entering the Bethe equations and the S-matrix [8–11]. Of course, Δ, Δ_{bare} and $E(g)$ may depend also on other quantum numbers, such as the spin chain length L, which also plays the role of a string angular momentum, other angular momenta, the Lorentz spin s, etc. Yet, the Beisert–Staudacher equations enjoy a validity seriously restricted by their scattering matrix origin, namely the length L and other quantum numbers need to be large. More precisely, starting from a certain loop order these equations are plagued by the so-called wrapping problem [12, 13]. Nevertheless, as scattering S-matrix equations [14], they are indeed correct and they can be interpreted as Bethe–Yang quantization conditions [15,16].

In quantum integrable 2D relativistic massive field theories, the problem of deriving off-shell quantities from on-shell information has been already addressed in many cases. For the purpose of this paper the derivation by Al B Zamolodchikov of the finite-size ground-state energy from the S-matrix [26] is relevant. Let us define the theory on a torus spacetime geometry. The space direction is finite with circumference L, time is periodic with period $R \to \infty$. Zamolodchikov's fascinating idea is to exchange space and time by defining a *mirror theory* in the infinite space R. In this mirror theory the space interval is infinite and the asymptotic Bethe–Yang equations hold true, but time is compact with size L. Now, we may interpret $L = 1/T$ as the inverse temperature and use the Yang–Yang thermodynamic Bethe ansatz (TBA) procedure [23] to find the minimum free energy or equivalently the ground-state energy for the (original) *direct* theory on a space circumference with size L. In the following, we will extend this procedure to the non-relativistic case relevant for the AdS/CFT correspondence.

We have been convinced that this strategy may be successful also in a complicated non-relativistic theory such as the AdS/CFT correspondence by the recent striking confirmation due to a sort of ancestor of the TBA for relativistic quantum field theory. In fact, Lüscher developed a method to compute, from scattering data, the finite-size corrections to the mass gap [17]. Later on, this method was specifically applied to integrable quantum field theories [18] and revealed itself as the leading term in the TBA large size expansion [29, 30]. Recently, a sophisticated extension of these ideas to the AdS/CFT correspondence has given striking results for the Konishi operator at four loops [19] and an impressive confirmation of the perturbative computations of [20].

In this paper, we will start from the equations recently formulated by Arutyunov and Frolov in [21] for the mirror theory of the $AdS_5 \times S^5$ superstring theory. These equations are derived by implementing the classification of all the particles and bound states in the Bethe–Yang equations derived in [22]. The classification is obtained with the formulation of the so-called string hypothesis of the Hubbard model (cf [24]): the map of the direct theory equations [15] into those of Hubbard's was already observed by Beisert [16]. Initially, we will modify the equations—in analogy with those of the Hubbard model [24]—so that we can take into account the information on the so-called $k - \Lambda$ strings. In this way, we produce a complete set of string equations for implementing the thermodynamic Bethe ansatz method and derive a set of TBA equations for the single particle dressed energies (the pseudoenergies). As a conclusion, the pseudoenergies determine the (free) energy via a nonlinear integral functional. We shall make explicit the similarity between our TBA equations and those for the Hubbard model and then derive a universal system of functional relations (the Y-system) for the exponential of the pseudoenergies. The universality of a Y-system consists in the fact that, at least for relativistic theories, it is the same for the excited states as well. Yet, there is by now a consolidated way towards excited states in relativistic massive field theories [29–31]. A very brief description of this procedure for the present case will be sketched in the final section, with the aim to gain a better control of the energy/dimension spectrum of the AdS_5/CFT_4 correspondence for any value of the coupling constant g and even for *short* operators. Apparently, the Y-system structure matches that recently proposed by Gromov, Kazakov and Vieira [37].

2. The equations for the root densities

As anticipated before, we need to pass from the $AdS_5 \times S^5$ theory defined on a circumference of length L to its mirror, and this has been extensively investigated by Arutyunov and Frolov since the paper [22]. In particular, they derive from the S-matrix the Bethe–Yang equations for the fundamental particles of the mirror theory:

$$\begin{aligned}
e^{i\widetilde{p}_k R} &= \prod_{\substack{l=1\\ l\neq k}}^{K^{\mathrm{I}}} (S_0(\widetilde{p}_k, \widetilde{p}_l))^2 \prod_{\alpha=1}^{2} \prod_{l=1}^{K^{\mathrm{II}}_{(\alpha)}} \frac{x_k^+ - y_l^{(\alpha)}}{x_k^- - y_l^{(\alpha)}} \sqrt{\frac{x_k^-}{x_k^+}}, \\
-1 &= \prod_{l=1}^{K^{\mathrm{I}}} \frac{y_k^{(\alpha)} - x_l^+}{y_k^{(\alpha)} - x_l^-} \sqrt{\frac{x_l^-}{x_l^+}} \prod_{l=1}^{K^{\mathrm{III}}_{(\alpha)}} \frac{v_k^{(\alpha)} - w_l^{(\alpha)} + \frac{i}{g}}{v_k^{(\alpha)} - w_l^{(\alpha)} - \frac{i}{g}}, \\
1 &= \prod_{l=1}^{K^{\mathrm{II}}_{(\alpha)}} \frac{w_k^{(\alpha)} - v_l^{(\alpha)} - \frac{i}{g}}{w_k^{(\alpha)} - v_l^{(\alpha)} + \frac{i}{g}} \prod_{\substack{l=1\\ l\neq k}}^{K^{\mathrm{III}}_{(\alpha)}} \frac{w_k^{(\alpha)} - w_l^{(\alpha)} + \frac{2i}{g}}{w_k^{(\alpha)} - w_l^{(\alpha)} - \frac{2i}{g}},
\end{aligned} \tag{2.1}$$

where

$$\left(S_0(\widetilde{p}_k, \widetilde{p}_l)\right)^2 = \frac{x_k^- - x_l^+}{x_k^+ - x_l^-} \frac{1 - \frac{1}{x_k^+ x_l^-}}{1 - \frac{1}{x_k^- x_l^+}} \sigma^2(x_k, x_l) \tag{2.2}$$

is the $a = 0$ light-cone gauge scalar factor of the mirror S-matrix, with $\sigma(x_k, x_l)$ the dressing factor in the mirror theory [22], and only here for the single particle case $x_k^\pm = x\left(u_k \pm \frac{i}{g}\right)$ (cf appendix A for the definition of the function $x(u)$). Thanks to a so-far formal resemblance of the last two BA equations (BAEs) with those of a inhomogeneous Hubbard model, they can formulate a string hypothesis for the solutions, in strict analogy with the Takahashi's one [24]. In few words, we assume that the thermodynamically relevant solutions[3] of (2.1) in the limit of large R, K^I, $K^{II}_{(\alpha)}$, $K^{III}_{(\alpha)}$ rearrange themselves into complexes—the so-called strings—with real centres and all the other complex roots symmetrically distributed around these centres along the imaginary direction. Paying attention to the presence of two coupled Hubbard models for $\alpha = 1, 2$, the strings may be classified as follows:

(1) N_Q Q-particles (bound states) with real momenta $\tilde{p}_k^Q$ and real rapidities u_k^Q:

$$u_k^{Q,j} = u_k^Q + (Q + 1 - 2j)\frac{\mathrm{i}}{g}, \qquad j = 1, \dots, Q; \tag{2.3}$$

(2) $N_y^{(\alpha)}$ $y^{(\alpha)}$-particles with real momenta $q_k^{(\alpha)}$;

(3) $N_{M|vw}^{(\alpha)}$ vw-strings with real centres v_k^M, $2M$ roots of type v and M of type w:

$$v_k^{M,j} = v_k^M \pm (M + 2 - 2j)\frac{\mathrm{i}}{g}, \qquad j = 1, \dots, M; \tag{2.4}$$

$$w_k^{M,j} = v_k^M + (M + 1 - 2j)\frac{\mathrm{i}}{g}, \qquad j = 1, \dots, M; \tag{2.5}$$

(4) $N_{N|w}^{(\alpha)}$ w-strings with real centres w_k^N and N roots of type w:

$$w_k^{N,j} = w_k^N + (N + 1 - 2j)\frac{\mathrm{i}}{g}, \qquad j = 1, \dots, N. \tag{2.6}$$

If the variables u_k, v_k and w_k in (2.1) are replaced by $u_k^{Q,j}$, $v_k^{M,j}$, $w_k^{M,j}$ and $w_k^{N,j}$, and the products on the internal string index j are made, then the equations for the real centres of the various kinds of the above strings (2.3)–(2.6) can be recast into the following form [21]:

$$1 = \mathrm{e}^{\mathrm{i}\tilde{p}_k^Q R} \prod_{Q'=1}^{\infty} \prod_{\substack{l=1\\ l\neq k}}^{N_{Q'}} S_{sl(2)}^{QQ'}(x_k, x_l) \prod_{\alpha=1}^{2} \prod_{l=1}^{N_y^{(\alpha)}} \frac{x_k^- - y_l^{(\alpha)}}{x_k^+ - y_l^{(\alpha)}} \sqrt{\frac{x_k^+}{x_k^-}} \prod_{M=1}^{\infty} \prod_{l=1}^{N_{M|vw}^{(\alpha)}} S_{xv}^{QM}\left(x_k, v_{l,M}^{(\alpha)}\right), \tag{2.7}$$

$$-1 = \prod_{Q=1}^{\infty} \prod_{l=1}^{N_Q} \frac{y_k^{(\alpha)} - x_l^+}{y_k^{(\alpha)} - x_l^-} \sqrt{\frac{x_l^-}{x_l^+}} \prod_{M=1}^{\infty} \prod_{l=1}^{N_{M|vw}^{(\alpha)}} \frac{v_k^{(\alpha)} - v_{l,M}^{(\alpha)-}}{v_k^{(\alpha)} - v_{l,M}^{(\alpha)+}} \prod_{N=1}^{\infty} \prod_{l=1}^{N_{N|w}^{(\alpha)}} \frac{v_k^{(\alpha)} - w_{l,N}^{(\alpha)-}}{v_k^{(\alpha)} - w_{l,N}^{(\alpha)+}}, \tag{2.8}$$

$$\prod_{Q=1}^{\infty} \prod_{l=1}^{N_Q} S_{xv}^{QK}\left(x_l, v_{k,K}^{(\alpha)}\right) = \prod_{M=1}^{\infty} \prod_{l=1}^{N_{M|vw}^{(\alpha)}} S_{vv}^{KM}\left(v_{k,K}^{(\alpha)}, v_{l,M}^{(\alpha)}\right) \prod_{N=1}^{\infty} \prod_{l=1}^{N_{N|w}^{(\alpha)}} S_{vw}^{KN}\left(v_{k,K}^{(\alpha)}, w_{l,N}^{(\alpha)}\right), \tag{2.9}$$

$$(-1)^K = \prod_{l=1}^{N_y^{(\alpha)}} \frac{w_{k,K}^{(\alpha)-} - v_l^{(\alpha)}}{w_{k,K}^{(\alpha)+} - v_l^{(\alpha)}} \prod_{N=1}^{\infty} \prod_{l=1}^{N_{N|w}^{(\alpha)}} S_{ww}^{KN}\left(w_{k,K}^{(\alpha)}, w_{l,N}^{(\alpha)}\right), \tag{2.10}$$

where, for the sake of shortness, notation changes from now on in that all the particle x-variables have to be read

$$x_k^{\pm} \equiv x_k^{Q\pm} = x\left(u_k^Q \pm \mathrm{i}\frac{Q}{g}\right), \tag{2.11}$$

[3] There is no definitive proof of the string hypothesis, though it seems to always give the correct thermodynamic limit. There might well be other kinds of solutions (which should not affect the thermodynamics).

and the definitions of the particle (i.e. subscripts k and l) variables $x(u)$, v, $v_K^\pm$ and $w_K^\pm$ (suppressing the index α) are listed in appendix A. The S-matrices are defined as follows:

$$S_{sl(2)}^{QQ'}(x_k,x_l)=\left[\prod_{j=1}^{Q}\prod_{h=1}^{Q'}\left(\frac{x(u_k+\mathrm{i}(Q+2-2j)/g)-x(u_l+\mathrm{i}(Q'-2h)/g)}{x(u_k+\mathrm{i}(Q-2j)/g)-x(u_l+\mathrm{i}(Q'+2-2h)/g)}\right)\right.$$
$$\left.\left(\frac{1-\frac{1}{x(u_k+\mathrm{i}(Q-2j)/g)x(u_l+\mathrm{i}(Q'+2-2h)/g)}}{1-\frac{1}{x(u_k+\mathrm{i}(Q+2-2j)/g)x(u_l+\mathrm{i}(Q'-2h)/g)}}\right)\right]\sigma_{Q,Q'}(u_k,u_l)^{-2}, \tag{2.12}$$

$$S_{xv}^{QM}(x_k,v_{l,M})=\left(\frac{x_k^{Q-}-x(v_{l,M}^+)}{x_k^{Q+}-x(v_{l,M}^+)}\right)\left(\frac{x_k^{Q-}-x(v_{l,M}^-)}{x_k^{Q+}-x(v_{l,M}^-)}\right)\left(\frac{x_k^{Q+}}{x_k^{Q-}}\right)$$
$$\times\prod_{j=1}^{M-1}\left(\frac{u_k-v_{l,M}-\mathrm{i}\frac{Q-M+2j}{g}}{u_k-v_{l,M}+\mathrm{i}\frac{Q-M+2j}{g}}\right),$$
$$S_{vv}^{KM}(x,y)=S_{vw}^{KM}(x,y)=S_{ww}^{KM}(x,y)=S_{KM}(x-y),$$
$$S_{KM}(u)=\left(\frac{u+\mathrm{i}\frac{|K-M|}{g}}{u-\mathrm{i}\frac{|K-M|}{g}}\right)\left(\frac{u+\mathrm{i}\frac{K+M}{g}}{u-\mathrm{i}\frac{K+M}{g}}\right)\prod_{k=1}^{\min(K,M)-1}\left(\frac{u+\mathrm{i}\frac{|K-M|+2k}{g}}{u-\mathrm{i}\frac{|K-M|+2k}{g}}\right)^2, \tag{2.13}$$

where $S_{(2)}^{QQ'}(x_k,x_l)$ is obtained from $\big(S_0(\widetilde{p}_k,\widetilde{p}_l)\big)^2$ by the fusion procedure [45, 46], with the upper index (length) of the string centre (Q in (2.3)) suppressed. Now, a simple crucial observation enters the stage: the last term on the rhs of (2.9) fails the resemblance with the usual Hubbard BAEs implemented by string hypothesis [24, 25]. In fact, we need one more step: we can easily see that the equation for the vw strings—corresponding to the Hubbard $k-\Lambda$ strings—do not have on the rhs a term of interaction between the w and vw strings; in contrast, there is a scattering term between a vw string and a single $v^{(\alpha)}$ (which do not belong to any string, but its own). Therefore, we may derive an intermediate equation

$$-1=\prod_{l=1}^{N_y^{(\alpha)}}\frac{w_k^{(\alpha)}-v_l^{(\alpha)}-\frac{\mathrm{i}}{g}}{w_k^{(\alpha)}-v_l^{(\alpha)}+\frac{\mathrm{i}}{g}}\prod_{N=1}^{\infty}\prod_{l=1}^{N_{N|w}^{(\alpha)}}\frac{w_k^{(\alpha)}-w_{l,N}^{(\alpha)-}+\frac{\mathrm{i}}{g}}{w_k^{(\alpha)}-w_{l,N}^{(\alpha)+}+\frac{\mathrm{i}}{g}}\,\frac{w_k^{(\alpha)}-w_{l,N}^{(\alpha)-}-\frac{\mathrm{i}}{g}}{w_k^{(\alpha)}-w_{l,N}^{(\alpha)+}-\frac{\mathrm{i}}{g}}, \tag{2.14}$$

and choose $w_k^{(\alpha)}$ belonging to a vw-string. With this little trick[4], we obtain

$$\prod_{N=1}^{\infty}\prod_{l=1}^{N_{N|w}^{(\alpha)}}S_{vw}^{KN}\big(v_{k,K}^{(\alpha)},w_{l,N}^{(\alpha)}\big)=(-1)^K\prod_{l=1}^{N_y^{(\alpha)}}\frac{v_{k,K}^{(\alpha)+}-v_l^{(\alpha)}}{v_{k,K}^{(\alpha)-}-v_l^{(\alpha)}}, \tag{2.15}$$

and finally we can rewrite (2.9) in a form re-echoing the Hubbard one:

$$\prod_{Q=1}^{\infty}\prod_{l=1}^{N_Q}S_{xv}^{QK}\big(x_l,v_{k,K}^{(\alpha)}\big)=\prod_{M=1}^{\infty}\prod_{l=1}^{N_{M|vw}^{(\alpha)}}S_{vv}^{KM}\big(v_{k,K}^{(\alpha)},v_{l,M}^{(\alpha)}\big)\prod_{N=1}^{\infty}\prod_{l=1}^{N_y^{(\alpha)}}S_{vy}^{K}\big(v_{k,K}^{(\alpha)},v_l^{(\alpha)}\big). \tag{2.16}$$

In (2.16) we have introduced a new scattering matrix

$$S_{vy}^{K}\big(v_{k,K}^{(\alpha)},v_l^{(\alpha)}\big)=\frac{v_{k,K}^{(\alpha)+}-v_l^{(\alpha)}}{v_{k,K}^{(\alpha)-}-v_l^{(\alpha)}}=\frac{v_{k,K}^{(\alpha)}-v_l^{(\alpha)}+\mathrm{i}K/g}{v_{k,K}^{(\alpha)}-v_l^{(\alpha)}-\mathrm{i}K/g}. \tag{2.17}$$

At this point, we can follow the standard TBA procedure [23–26], which goes in a very sketchy way as follows. After taking the logarithm of these equations, we shall consider the

[4] After the first version of this paper appeared on the arXiv, this trick was implemented in a revised version of [21].

thermodynamic limit (K^I, $N_y^{(\alpha)}$, $N_{vw}^{(\alpha)}$, $N_w^{(\alpha)}$, $R \to \infty$) while keeping the densities finite (sums of root and hole densities, respectively)

$$\rho_Q(\tilde{p}) = \rho_Q^r(\tilde{p}) + \rho_Q^h(\tilde{p}) = \lim_{R\to\infty} \frac{I_{k+1}^Q - I_k^Q}{R\big(\tilde{p}_{k+1}^Q - \tilde{p}_k^Q\big)}, \tag{2.18}$$

$$\rho_y^\alpha(q) = \rho_y^{r\alpha}(q) + \rho_y^{h\alpha}(q) = \lim_{R\to\infty} \frac{I'^\alpha_{k+1} - I'^\alpha_k}{R\big(q_{k+1}^{(\alpha)} - q_k^{(\alpha)}\big)}, \tag{2.19}$$

$$\rho_{v,K}^\alpha(\lambda) = \rho_{v,K}^{r\alpha}(\lambda) + \rho_{v,K}^{h\alpha}(\lambda) = \lim_{R\to\infty} \frac{J_{k+1}^{K^\alpha} - J_k^{K^\alpha}}{R\big(\lambda_{k+1}^{(\alpha)} - \lambda_k^{(\alpha)}\big)}, \tag{2.20}$$

$$\rho_{w,K}^\alpha(\lambda) = \rho_{w,K}^{r\alpha}(\lambda) + \rho_{w,K}^{h\alpha}(\lambda) = \lim_{R\to\infty} \frac{J'^{K^\alpha}_{k+1} - J'^{K^\alpha}_k}{R\big(\lambda_{k+1}^{(\alpha)} - \lambda_k^{(\alpha)}\big)}, \tag{2.21}$$

where the Is and the Js are the integer and half-integer quantum numbers, respectively. Eventually, we can produce for the thermodynamic state the following integral equations constraining the densities:

$$\rho_Q(\tilde{p}) = \frac{1}{2\pi} + \sum_{Q'=1}^{\infty} \big(\phi_{sl(2)}^{QQ'} * \rho_{Q'}^r\big)(\tilde{p}) + \sum_{\alpha=1}^{2} \left[\big(\phi_{xy}^Q * \rho_y^{r\alpha}\big) + \sum_{M=1}^{\infty} \big(\phi_{xv}^{QM} * \rho_{v,M}^{r\alpha}\big)\right](\tilde{p}), \tag{2.22}$$

$$\rho_y^\alpha(q) = \sum_{Q=1}^{\infty} \big(\phi_{yx}^Q * \rho_Q^r\big)(q) + \sum_{M=1}^{\infty} \big(\phi_{yv}^M * \rho_{v,M}^{r\alpha}\big)(q) + \sum_{N=1}^{\infty} \big(\phi_{yw}^N * \rho_{w,N}^{r\alpha}\big)(q), \tag{2.23}$$

$$\rho_{v,K}^\alpha(\lambda) = \sum_{M=1}^{\infty} \big(\phi_{vv}^{KM} * \rho_{v,M}^{r\alpha}\big)(\lambda) + \big(\phi_{vx}^{KQ} * \rho_Q^r\big)(\lambda) + \big(\phi_{vy}^K * \rho_y^{r\alpha}\big)(\lambda), \tag{2.24}$$

$$\rho_{w,K}^\alpha(\lambda) = \sum_{M=1}^{\infty} \big(\phi_{ww}^{KM} * \rho_{w,M}^{r\alpha}\big)(\lambda) + \big(\phi_{wy}^K * \rho_y^{r\alpha}\big)(\lambda), \tag{2.25}$$

where the symbol $*$ denotes the *usual* convolution (on the second variable) $(\phi * g)(z) = \int \mathrm{d}z' \phi(z,z') g(z')$ and the kernels are defined in appendix A[5].

3. The thermodynamic Bethe ansatz equations

We continue our very sketchy presentation of the derivation of the TBA equations. For this purpose, we express the entropy in terms of the hole and root densities[6] (ρ^h and ρ^r, respectively)

$$S = \sum_{Q=1}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}\tilde{p}\, \big(\big[\rho_Q^r(\tilde{p}) + \rho_Q^h(\tilde{p})\big] \ln\big[\rho_Q^r(\tilde{p}) + \rho_Q^h(\tilde{p})\big] - \rho_Q^r(\tilde{p}) \ln \rho_Q^r(\tilde{p}) - \rho_Q^h(\tilde{p}) \ln \rho_Q^h(\tilde{p})\big)$$

$$+ \sum_{\alpha=1}^{2} \int_{-\pi}^{\pi} \mathrm{d}q\, \big(\big[\rho_y^{r\alpha}(q) + \rho_y^{h\alpha}(q)\big] \ln\big[\rho_y^{r\alpha}(q) + \rho_y^{h\alpha}(q)\big] - \rho_y^{r\alpha}(q) \ln \rho_y^{r\alpha}(q) - \rho_y^{h\alpha}(q) \ln \rho_y^{h\alpha}(q)\big)$$

[5] We begin to note that here the kernels $\phi(z, z')$ do not necessarily depend on the difference $(z - z')$.

[6] Hereafter the integration measure $\mathrm{d}\tilde{p}$ has to be interpreted as the Stieltjes measure $\frac{\mathrm{d}\tilde{p}}{\mathrm{d}u}\mathrm{d}u$, as $\tilde{p}$ depends on (the parameters) Q and g as well.

$$+ \sum_{\alpha=1}^{2}\sum_{M=1}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}\lambda\big(\big[\rho_{v,M}^{r\alpha}(\lambda)+\rho_{v,M}^{h\alpha}(\lambda)\big]\ln\big[\rho_{v,M}^{r\alpha}(\lambda)+\rho_{v,M}^{h\alpha}(\lambda)\big] - \rho_{v,M}^{r\alpha}(\lambda)\ln\rho_{v,M}^{r\alpha}(\lambda) - \rho_{v,M}^{h\alpha}(\lambda)\ln\rho_{v,M}^{h\alpha}(\lambda)\big)$$

$$+ \sum_{\alpha=1}^{2}\sum_{N=1}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}\lambda\big(\big[\rho_{w,N}^{r\alpha}(\lambda)+\rho_{w,N}^{h\alpha}(\lambda)\big]\ln\big[\rho_{w,N}^{r\alpha}(\lambda)+\rho_{w,N}^{h\alpha}(\lambda)\big] - \rho_{w,N}^{r\alpha}(\lambda)\ln\rho_{w,N}^{r\alpha}(\lambda) - \rho_{w,N}^{h\alpha}(\lambda)\ln\rho_{w,N}^{h\alpha}(\lambda)\big), \tag{3.1}$$

and then minimize the free energy per unit length,

$$f(T) = \tilde{H} - TS, \tag{3.2}$$

where $\tilde{H}$ is the mirror energy per unit length [22]:

$$\tilde{H} = 2\sum_{Q=1}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}\tilde{p}\ \ \mathrm{arcsinh}\left(\frac{\sqrt{Q^2+\tilde{p}^2}}{2g}\right)\rho_Q^r(\tilde{p}). \tag{3.3}$$

As stated before, then we ought to treat the temperature T of the mirror theory as the inverse of the size L in the AdS/CFT: $T = 1/L$. The extremum condition $\delta f = 0$ under the constraints (2.22)–(2.25) entails the final set of thermodynamic Bethe ansatz equations for the pseudoenergies ϵ_A such that

$$\epsilon_A = \ln\frac{\rho_A^h}{\rho_A^r}, \qquad \frac{1}{\mathrm{e}^{\epsilon_A}+1} = \frac{\rho_A^r}{\rho_A}, \qquad L_A = \ln(1+\mathrm{e}^{-\epsilon_A}), \tag{3.4}$$

with the short indication of the collective index A for the different density labels. The ground-state thermodynamic Bethe ansatz equations are

$$\epsilon_Q(\tilde{p}) = 2L\ \ \mathrm{arcsinh}\left(\frac{\sqrt{Q^2+\tilde{p}^2}}{2g}\right) - \sum_{Q'=1}^{\infty}\big(\phi_{sl(2)}^{Q'Q} * L_{Q'}\big)(\tilde{p}) - \sum_{\alpha=1}^{2}\big(\phi_{yx}^{Q} * L_y^\alpha\big)(\tilde{p}) - \sum_{\alpha=1}^{2}\sum_{M=1}^{\infty}\big(\phi_{vx}^{MQ} * L_{v,M}^\alpha\big)(\tilde{p}), \tag{3.5}$$

$$\epsilon_y^\alpha(q) = -\sum_{Q=1}^{\infty}\big(\phi_{xy}^{Q} * L_Q\big)(q) - \sum_{M=1}^{\infty}\big(\phi_{wy}^{M} * L_{w,M}^\alpha\big)(q) - \sum_{N=1}^{\infty}\big(\phi_{vy}^{N} * L_{v,N}^\alpha\big)(q), \tag{3.6}$$

$$\epsilon_{v,K}^\alpha(\lambda) = -\sum_{Q=1}^{\infty}\big(\phi_{xv}^{QK} * L_Q\big)(\lambda) - \big(\phi_{yv}^{K} * L_y^\alpha\big)(\lambda) - \sum_{M=1}^{\infty}\big(\phi_{vv}^{MK} * L_{v,M}^\alpha\big)(\lambda), \tag{3.7}$$

$$\epsilon_{w,K}^\alpha(\lambda) = -\big(\phi_{yw}^{K} * L_y^\alpha\big)(\lambda) - \sum_{M=1}^{\infty}\big(\phi_{ww}^{MK} * L_{w,M}^\alpha\big)(\lambda), \tag{3.8}$$

with $\alpha = 1, 2$, $Q = 1, 2, \ldots$ and $K = 1, 2, \ldots$. Note that, apart from the specific form of the kernels (see appendix A for their definitions), the TBA equations are similar in form to the density equations (2.22)–(2.25), provided we exchange $\rho \to -L$. However, we should stress that on our way from (2.22)–(2.25) to (3.5)–(3.8) we have made an abuse of notation and changed definition for the convolution $*$ moving on to the first variable

$$(\phi * g)(z) = \int \mathrm{d}z' \phi(z', z) g(z'). \tag{3.9}$$

When the kernel $\phi(z, z')$ can be written only as an (even) function of the difference $(z - z')$, as for example in the relativistic theories discussed in [26, 28], this change in the definition of $*$ can be avoided by keeping the convolution on the second variable. However, in the present framework some of the kernels have a genuinely different functional dependence on the two independent variables and this simplification is absent. Moreover, an important comment is here on the integration limits: they are from $-\infty$ to ∞ for the λ- and $\tilde{p}$-variables but from $-\pi$ to π for the q-variables.

As a concluding result, the *minimal* free energy for the mirror theory results by inserting the TBA equations into the general (3.2) and is given by the following nonlinear functional of the pseudoenergies $\epsilon_Q(u)$:

$$f(T) = -T \sum_{Q=1}^{\infty} \int_{-\infty}^{\infty} \frac{\mathrm{d}\tilde{p}}{2\pi} \ln(1 + \mathrm{e}^{-\epsilon_Q(\tilde{p})}) = -T \sum_{Q=1}^{\infty} \int_{-\infty}^{\infty} \frac{\mathrm{d}u}{2\pi} \frac{\mathrm{d}\tilde{p}}{\mathrm{d}u} \ln(1 + \mathrm{e}^{-\epsilon_Q(u)}). \tag{3.10}$$

Consequently, the ground-state energy for the AdS/CFT theory on a circumference with length $L = 1/T$ ought to satisfy the relation

$$E_0(L) = Lf(1/L). \tag{3.11}$$

As we have kept the total densities finite, it is natural to introduce chemical potentials μ_A. This has been already finalized in relativistic theories by [28]. The TBA equations (3.5)–(3.8) do not change their form, but for this simple replacement

$$L_A = \ln(1 + \mathrm{e}^{-\epsilon_A}) \to L_{A,\lambda} = \ln(1 + \lambda_A \mathrm{e}^{-\epsilon_A}), \tag{3.12}$$

involving the fugacities $\lambda_A = \mathrm{e}^{\mu_A/T}$. Here, we would like to conjecture that their introduction should be related to the zero energy of the ground state (independently of the value of T) which is a half BPS protected state. It is a consequence of a result by [33], further developed in [32] and in [34] that in particular $\mathcal{N} = 2$ supersymmetric theories this size-invariant state can be selected via a suitable tuning of the TBA fugacities. A plot describing this interesting transition, as the fugacities approach these critical values, can be found in [35]. In our case we expect zero energy as soon as the fugacities reach these values:

$$\lambda_Q = 1, \quad \lambda_{v,K}^{\alpha} = -1, \quad \lambda_{w,K}^{\alpha} = (-1)^{K+1}, \quad \lambda_y^{\alpha} = -1 \quad (\alpha = 1, 2, K = 1, 2, \ldots). \tag{3.13}$$

Physically, this modification corresponds to the calculation of the Witten index. In (3.13), the fermionic and bosonic character of the pseudoparticles is chosen following an analogy with other scattering-matrix models and considering the evident Z_2-symmetry of the TBA equations. There are, of course, other possibilities. The vanishing of ground-state energy in TBA models is a very delicate issue and we prefer to postpone this discussion to the near future and in presence of analytic or numerical evidences.

3.1. A comparison with the Hubbard TBA equations

As the reader can see in appendix A, some kernels in (3.5)–(3.8) actually depend on the difference of rapidities. Therefore, the convolution involving these kernels is a standard 'difference' convolution, i.e. $(f * g)(z) = \int \mathrm{d}z' f(z - z') g(z')$. In other words, we may rewrite equations (3.6)–(3.8) in a form that is closer to the TBA equations of the Hubbard model, as we might expect from the analogy at the level of Bethe ansatz equations. Of course, we must leave untouched the terms really depending on the two different variables and think of them as *driving* or *forcing* terms connecting the two Hubbard models. For this reason, we move them on the lhs of the equations and write

$$\epsilon_y^\alpha(q) + \sum_{Q=1}^{\infty}\left(\phi_{xy}^{Q} * L_Q\right)(q) = \sum_{M=1}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}\lambda\, a_M(\lambda - \sin(q)) \ln\left(1 + \mathrm{e}^{-\epsilon_{v,M}^{\alpha}(\lambda)}\right) - \sum_{M=1}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}\lambda\, a_M(\lambda - \sin(q)) \ln(1 + \mathrm{e}^{-\epsilon_{w,M}(\lambda)}), \tag{3.14}$$

$$\epsilon_{v,K}^\alpha(\lambda) + \sum_{Q=1}^{\infty}\left(\phi_{xv}^{QK} * L_Q\right)(\lambda) = -\int_{-\pi}^{\pi} \mathrm{d}q \cos(q) a_K(\sin(q) - \lambda) \ln\left(1 +^{-\epsilon_y^\alpha(q)}\right) + \sum_{M=1}^{\infty}\left(A_{MK} * L_{v,M}^{\alpha}\right)(\lambda), \tag{3.15}$$

$$\epsilon_{w,K}^\alpha(\lambda) = -\int_{-\pi}^{\pi} \mathrm{d}q \cos(q) a_K(\sin(q) - \lambda) \ln\left(1 + \mathrm{e}^{-\epsilon_y^\alpha(q)}\right) + \sum_{M=1}^{\infty}\left(A_{MK} * L_{w,M}^{\alpha}\right)(\lambda), \tag{3.16}$$

where

$$a_K(x) = \frac{1}{2\pi}\frac{K/g}{(K/2g)^2 + x^2}, \tag{3.17}$$

$$(A_{MK} * L)(x) = \int_{-\infty}^{\infty} \frac{\mathrm{d}y}{2\pi}\frac{\mathrm{d}}{\mathrm{d}x}\Theta_{MK}\left(2g(x-y)\right) L(y), \tag{3.18}$$

$$\Theta_{MK}(x) = \begin{cases} \theta\left(\frac{x}{|K-M|}\right) + 2\theta\left(\frac{x}{|K-M|+2}\right) + \cdots \\ + 2\theta\left(\frac{x}{K+M-2}\right) + \theta\left(\frac{x}{K+M}\right), & \text{if } K \neq M \\ 2\theta\left(\frac{x}{2}\right) + 2\theta\left(\frac{x}{4}\right) + \cdots + 2\theta\left(\frac{x}{2M-2}\right) + \theta\left(\frac{x}{2M}\right), & \text{if } K = M, \end{cases} \tag{3.19}$$

$$\theta(x) = 2\arctan(x). \tag{3.20}$$

Equations (3.14)–(3.20) should be compared with equations (5.43) and (5.54)–(5.56) in [25] evaluated at $\bar{u} \equiv u^{\text{Ref.[25]}} = 1/2g$.

In the following sections we shall derive a set of functional identities (Y-system) satisfied by the quantities $Y_A = \mathrm{e}^{\epsilon_A}$ (or $= \mathrm{e}^{-\epsilon_A}$). Very importantly, a Y-system is universal in the sense that it is the same for *all* the energy states $E_n(L)$, at least in a relativistic theory [29, 30]. Fugacities as those defined in (3.13) may be removed by a simple redefinition of the Y's. Therefore, these are discharged in the next sections.

4. Y-system for the Hubbard model

The TBA equations for the Hubbard model in universal form are written, for example, in [25][7]. This section is not meant to be particularly original and its aim is to explain how a subset of the Y-system equations proposed in [37] and in this paper emerges from the Hubbard model. The TBA equations are as follows:

$$\begin{aligned}
\ln\eta_1(\lambda) &= s*\ln(1+\eta_2)(\lambda) - \int_{-\pi}^{\pi} \mathrm{d}k\cos(k)s(\lambda-\sin(k))\ln\left(1+\frac{1}{\zeta(k)}\right),\\
\ln\eta_1'(\lambda) &= s*\ln(1+\eta_2')(\lambda) - \int_{-\pi}^{\pi} \mathrm{d}k\cos(k)s(\lambda-\sin(k))\ln(1+\zeta(k)),\\
\ln\eta_n(\lambda) &= s*\ln[(1+\eta_{n-1})(1+\eta_{n+1})](\lambda), \qquad n=2,3,\ldots,\\
\ln\eta_n'(\lambda) &= s*\ln[(1+\eta_{n-1}')(1+\eta_{n+1}')](\lambda), \qquad n=2,3,\ldots,
\end{aligned} \tag{4.1}$$

and

$$\begin{aligned}
\ln\zeta(k) = -\frac{2}{T}\cos(k) - \frac{1}{T}\int_{-\infty}^{\infty}\mathrm{d}\lambda s(\sin(k)-\lambda)(4\mathrm{Re}\sqrt{1-(\lambda-\mathrm{i}\bar{u})^2})\\
+\int_{-\infty}^{\infty}\mathrm{d}y s(\sin(k)-\lambda)\ln\left(\frac{1+\eta_1'}{1+\eta_1}\right),
\end{aligned} \tag{4.2}$$

where

$$s(\lambda) = \frac{1}{4\bar{u}\cosh(\pi\lambda/2\bar{u})} \tag{4.3}$$

is the convolution kernel. $s(\lambda)$ fulfils the following important property:

$$s(\lambda+\mathrm{i}\bar{u}) + s(\lambda-\mathrm{i}\bar{u}) = \delta(\lambda). \tag{4.4}$$

Relation (4.4) leads to the following set of functional relations:

$$\eta_n(\lambda+\mathrm{i}\bar{u})\eta_n(\lambda-\mathrm{i}\bar{u}) = (1+\eta_{n-1}(\lambda))(1+\eta_{n+1}(\lambda)), \tag{4.5}$$

$$\eta_n'(\lambda+\mathrm{i}\bar{u})\eta_n'(\lambda-\mathrm{i}\bar{u}) = (1+\eta_{n-1}'(\lambda))(1+\eta_{n+1}'(\lambda)), \tag{4.6}$$

with $n = 2, 3, \ldots$. For $n = 1$ we have instead

$$\ln[\eta_1(\lambda+\mathrm{i}\bar{u})\eta_1(\lambda-\mathrm{i}\bar{u})] = \ln[(1+\eta_2)(\lambda)] - \int_{-\pi}^{\pi}\mathrm{d}k\cos(k)\delta(\lambda-\sin(k))\ln\left(1+\frac{1}{\zeta(k)}\right),$$

$$\ln[\eta_1'(\lambda+\mathrm{i}\bar{u})\eta_1'(\lambda-\mathrm{i}\bar{u})] = \ln[(1+\eta_2')(\lambda)] - \int_{-\pi}^{\pi}\mathrm{d}k\cos(k)\delta(\lambda-\sin(k))\ln(1+\zeta(k)).$$

But for fixed $0 < \lambda < 1$, the argument of the Dirac δ function vanishes twice, i.e. at $k = \arcsin(\lambda)$ and $k = \pi - \arcsin(\lambda)$. This gives

$$\eta_1(\lambda+\mathrm{i}\bar{u})\eta_1(\lambda-\mathrm{i}\bar{u}) = (1+\eta_2(\lambda))\left(\frac{1+1/\zeta(\pi-k)}{1+1/\zeta(k)}\right), \tag{4.7}$$

$$\eta_1'(\lambda+\mathrm{i}\bar{u})\eta_1'(\lambda-\mathrm{i}\bar{u}) = (1+\eta_2'(\lambda))\left(\frac{1+\zeta(\pi-k)}{1+\zeta(k)}\right). \tag{4.8}$$

Finally considering that $\cos(k) = -\sqrt{1-\sin^2(k)}$ for $\pi/2 < k < \pi$, we get

$$\zeta^+(\pi-k)\zeta^-(\pi-k) \equiv \zeta(\pi-\arcsin(\lambda+\mathrm{i}\bar{u}))\zeta(\pi-\arcsin(\lambda-\mathrm{i}\bar{u})) = \left(\frac{1+\eta_1'(\lambda)}{1+\eta_1(\lambda)}\right). \tag{4.9}$$

[7] See also [36] for the Y-system and the excited states in a closely related model.

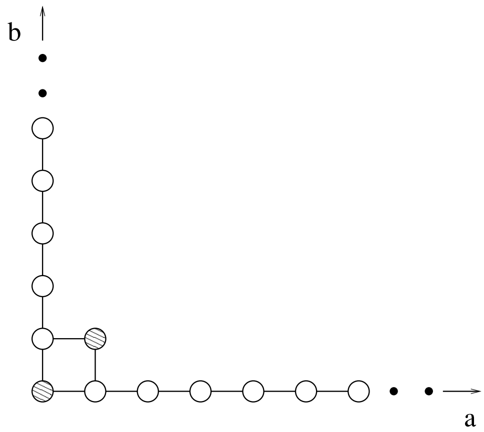

Figure 1. The Hubbard diagram.

From the relation

$$\zeta(\pi - k) = \zeta(k)e^{4\cos(k)/T} \tag{4.10}$$

(see equation (5.A.2) in [25]) we also have

$$\zeta^+(k)\zeta^-(k) \equiv \zeta(\arcsin(\lambda + i\bar{u}))\zeta(\arcsin(\lambda - i\bar{u})) = \left(\frac{1 + \eta_1'(\lambda)}{1 + \eta_1(\lambda)}\right) \times e^{\frac{4}{T}(\sqrt{1-(\sin(k)+i\bar{u})^2}+\sqrt{1-(\sin(k)-i\bar{u})^2})}. \tag{4.11}$$

To see the relationship with the Y-system represented in figure 1 of [37], set $z_i = 1/\eta_i'$:

$$z_1(\lambda + i\bar{u})z_1(\lambda - i\bar{u}) = (1 + 1/z_2(\lambda))^{-1}\left(\frac{1 + \zeta(k)}{1 + \zeta(\pi - k)}\right), \tag{4.12}$$

$$z_n(\lambda + i\bar{u})z_n(\lambda - i\bar{u}) = (1 + 1/z_{n-1}(\lambda))^{-1}(1 + 1/z_{n+1}(\lambda))^{-1}, \tag{4.13}$$

and

$$Y_{22}(k) = \zeta(k), \qquad Y_{11}(k) \equiv 1/Y_{22}(\pi - k) = 1/\zeta(\pi - k), \tag{4.14}$$

$$Y_{1,b+1}(\lambda) = z_b(\lambda), \qquad Y_{a+1,1}(\lambda) = \eta_a(\lambda), \tag{4.15}$$

with $a, b = 1, 2, 3, \ldots$ and construct a TBA diagram using the following rules [38]:

- starting from a given node (a, b) the lhs of the Y-system is always $Y_{ab}(\lambda + i\bar{u})Y_{ab}(\lambda - i\bar{u})$;
- a horizontal link between the nodes (a, b) and (a', b) corresponds to a factor $(1 + Y_{a'b}(\lambda))$ on the rhs;
- a vertical link between (a, b) and (a, b') corresponds to a factor $(1 + 1/Y_{ab'}(\lambda))^{-1}$ on the rhs.

It is easy to check that the diagram represented in figure 1 is reproduced with the exception of the functional relation (4.11) for $Y_{22}(\lambda(k)) = \zeta(k)$ which would close a 'standard' Y-system diagram only if this extra constraint were true:

$$\frac{\eta_1(\lambda(k))}{\eta_1'(\lambda(k))} = e^{\frac{4}{T}(\sqrt{1-(\sin(k)+i\bar{u})^2}+\sqrt{1-(\sin(k)-i\bar{u})^2})}. \tag{4.16}$$

This equation certainly holds at $T = \infty$ and would be compatible with some of the evident symmetries of the TBA equations but still it would imply a chain of extra constraints (on the other TBA functions) that we did not try to prove. In fact, we should stress that we have included the node Y_{11}, which is related to Y_{22} by (4.14) and (4.10). Therefore, there is no need to show an extra equation for $Y_{22}(\lambda(k)) = \zeta(k)$[8] once we already have $\ln Y_{11}(\lambda(k)) = -\ln\zeta(\pi - k)$ in the TBA system.

5. Y-system for the AdS/CFT correspondence

Let us start from equation (3.8) and observe that $S_{KM}(u)$ defined in (2.13) are a particular $n \to \infty$ limit of the Z_n-related scattering matrix elements proposed in [39]. They satisfy the following set of functional relations [27, 40]:

$$S_{KM}\left(2\lambda + \frac{\mathrm{i}}{g}\right) S_{KM}\left(2\lambda - \frac{\mathrm{i}}{g}\right) = \prod_{K'=1}^{\infty} (S_{K'M}(2\lambda))^{I_{KK'}}\, \mathrm{e}^{-\mathrm{i}2\pi I_{KM}\Theta(2\lambda)}, \tag{5.1}$$

where $I_{NM} = \delta_{N,M+1} + \delta_{N,M-1}$ and $\Theta(u)$ is the Heaviside step function. Equation (5.1) leads to

$$\phi_{ww}^{KM}\left(\lambda' - \lambda + \frac{\mathrm{i}}{2g}\right) + \phi_{ww}^{KM}\left(\lambda' - \lambda - \frac{\mathrm{i}}{2g}\right) - \sum_{K'=1}^{\infty} I_{KK'}\phi_{ww}^{K'M}(\lambda'-\lambda) = -I_{KM}\delta(\lambda'-\lambda). \tag{5.2}$$

Note that $\phi_{ww}^{KM}(\lambda)$ is equal to $-A_{KM}(\lambda)$ defined in equation (3.18). Another relevant identity is

$$\begin{aligned}&\phi_{yw}^{K}\left(\sin(q'), \lambda + \frac{\mathrm{i}}{2g}\right) + \phi_{yw}^{K}\left(\sin(q'), \lambda - \frac{\mathrm{i}}{2g}\right) - \sum_{K'=1}^{\infty} I_{KK'}\phi_{yw}^{K'}(\sin(q'), \lambda)\\ &= -\delta_{K1}\cos(q')\delta(\sin(q') - \lambda).\end{aligned} \tag{5.3}$$

Using equations (5.2), (5.3) and setting

$$Y_{w,K}^{\alpha}(\lambda) = \mathrm{e}^{-\epsilon_{w,K}^{\alpha}(\lambda)}, \qquad Y_{y}^{\alpha}(q) = \mathrm{e}^{\epsilon_{y}^{\alpha}(q)}, \qquad Y_{y^*}^{\alpha}(q) \equiv \mathrm{e}^{\epsilon_{y^*}^{\alpha}(q)} = \mathrm{e}^{-\epsilon_{y}^{\alpha}(\pi - q)}, \tag{5.4}$$

with $q = \arcsin(\lambda)$, we find

$$Y_{w,K}^{\alpha}\left(\lambda + \frac{\mathrm{i}}{2g}\right) Y_{w,K}^{\alpha}\left(\lambda - \frac{\mathrm{i}}{2g}\right) = \prod_{K'=1}^{\infty}\left(1 + \frac{1}{Y_{w,K'}^{\alpha}(\lambda)}\right)^{-I_{KK'}} \left(\frac{1 + Y_{y^*}^{\alpha}(q)}{1 + 1/Y_{y}^{\alpha}(q)}\right)^{\delta_{K1}}. \tag{5.5}$$

Let us now consider equation (3.7). The identity (B.8) with $K = 2, 3, \ldots$, together with equations (5.2) and (5.3) leads to

$$Y_{v,K}^{\alpha}\left(\lambda + \frac{\mathrm{i}}{2g}\right) Y_{v,K}^{\alpha}\left(\lambda - \frac{\mathrm{i}}{2g}\right) = \prod_{K'=1}^{\infty} \left(1 + Y_{v,K'}^{\alpha}(\lambda)\right)^{I_{KK'}} \left(1 + \frac{1}{Y_{K+1}(\tilde{p})}\right)^{-1}, \tag{5.6}$$

with $\tilde{p} = \tilde{p}(2\lambda)$ defined in (A.10) and $Y_{v,K}^{\alpha} = \mathrm{e}^{\epsilon_{v,K}^{\alpha}}$. The case with $K = 1$ is slightly more tricky, but the strategy is just the same. One starts considering the expression

$$\epsilon_{v1}^{\alpha}\left(\lambda + \frac{\mathrm{i}}{2g}\right) + \epsilon_{v1}^{\alpha}\left(\lambda - \frac{\mathrm{i}}{2g}\right) - \epsilon_{v2}^{\alpha}(\lambda) - \epsilon_{y}^{\alpha}(q) - \epsilon_{y^*}^{\alpha}(q), \tag{5.7}$$

with $q = \arcsin(\lambda)$. The corresponding rhs of the TBA equations cancel almost completely due to the functional relations fulfilled by the kernel functions; they just leave some 'contact' delta function contributions. The following identities are useful:

$$2\pi \mathrm{i}\phi_1(u, v) = \frac{\mathrm{d}}{\mathrm{d}u}\ln[(x(u) - x(v))(1 - x^{-1}(u)x^{-1}(v))] = \frac{1}{u - v}, \tag{5.8}$$

[8] *A fortiori*, if this equation should not respect the 'standard' form of the *Y*-system.

$$2\pi i\phi_2(u,v) = \frac{d}{du}\ln\left[\frac{x(u)-x(v)}{x(u)-x^{-1}(v)}\right] = \frac{\sqrt{4-v^2}}{\sqrt{4-u^2}}\frac{1}{u-v}, \tag{5.9}$$

$$\frac{1}{2\pi i}\frac{d}{du}\ln\left[\frac{(x(u)-x(v))^2}{x(u)}\right] = \phi_1(u,v)+\phi_2(u,v), \tag{5.10}$$

$$\phi^Q(u,v) = \phi_2\left(u - i\frac{Q}{g}, v\right) - \phi_2\left(u + i\frac{Q}{g}, v\right), \tag{5.11}$$

and also the equation

$$\epsilon_y^\alpha(q) + \epsilon_{y^*}^\alpha(q) \equiv \epsilon_y^\alpha(q) - \epsilon_y^\alpha(\pi - q) = -\sum_{Q=1}^{\infty}\phi^Q * L_Q(q), \tag{5.12}$$

analogous to (4.10) for the Hubbard model. The result is

$$Y_{v,1}^\alpha\left(\lambda+\frac{i}{2g}\right)Y_{v,1}^\alpha\left(\lambda-\frac{i}{2g}\right) = \left(1+Y_{v,2}^\alpha(\lambda)\right)\left(1+Y_y^\alpha(q)\right) \times \left(1+\frac{1}{Y_{y^*}(q)}\right)^{-1}\left(1+\frac{1}{Y_2(\tilde{p})}\right)^{-1}. \tag{5.13}$$

Further, consider the quantity

$$\epsilon_y(q^+) + \epsilon_y(q^-) - \epsilon_{v,1}(\lambda), \tag{5.14}$$

where $q^\pm = \arcsin(\lambda \pm i/2g)$; the kernel properties and the TBA equation (3.8) at $K = 1$ give

$$Y_y^\alpha(q^+)Y_y^\alpha(q^-) = \left(1+Y_{v,1}^\alpha(\lambda)\right)\left(1+\frac{1}{Y_{w,1}^\alpha(\lambda)}\right)^{-1}\left(1+\frac{1}{Y_1(\tilde{p})}\right)^{-1}, \tag{5.15}$$

with $\tilde{p} = \tilde{p}(2\lambda)$. Finally, using the property

$$\frac{x^{Q+}(u+i/g)}{x^{Q-}(u+i/g)}\frac{x^{Q+}(u-i/g)}{x^{Q-}(u-i/g)} = \frac{x^{(Q-1)+}(u)}{x^{(Q-1)-}(u)}\frac{x^{(Q+1)+}(u)}{x^{(Q+1)-}(u)}, \tag{5.16}$$

and similar relations for $\phi_{sl(2)}^{Q'Q}$, ϕ_{yx}^Q and ϕ_{vx}^{QM} (see appendix B), we get

$$Y_Q\left(x\left(u+\frac{i}{g}\right)\right)Y_Q\left(x\left(u-\frac{i}{g}\right)\right) = \prod_{Q'=1}^{\infty}(1+Y_{Q'}(x(u)))^{I_{QQ'}}\prod_{\alpha=1}^{2}\left(1+\frac{1}{Y_{v,Q-1}^\alpha(\lambda)}\right)^{-1}, \tag{5.17}$$

with $Q = 2, 3, \ldots$ and

$$Y_1\left(x\left(u+\frac{i}{g}\right)\right)Y_1\left(x\left(u-\frac{i}{g}\right)\right) = (1+Y_2(x(u)))\prod_{\alpha=1}^{2}\left(1+\frac{1}{Y_y^\alpha(q)}\right)^{-1}, \tag{5.18}$$

with $q = \arcsin(\lambda)$, $u = 2\lambda$ and $Y_Q = e^{\epsilon_Q}$. Setting

$$Y_{Q,0} = Y_Q, \qquad Y_{1,1} = Y_y^1, \qquad Y_{1,-1} = Y_y^2, \qquad Y_{2,2} = Y_{y^*}^1, \qquad Y_{2,-2} = Y_{y^*}^2,$$
$$Y_{1,K+1} = Y_{w,K}^1, \qquad Y_{1,-K-1} = Y_{w,K}^2, \qquad Y_{K+1,1} = Y_{v,K}^1, \qquad Y_{K+1,-1} = Y_{v,K}^2, \tag{5.19}$$

and following the rules given at the end of section 4 we may encode this Y-system in the diagram shown in figure 2. In other words, equations (5.5)–(5.18) with the identifications (5.19) can be recast in the compact form:

$$Y_{a,b}^+Y_{a,b}^- = (1+Y_{a+1,b})(1+Y_{a-1,b})\left(1+\frac{1}{Y_{a,b+1}}\right)^{-1}\left(1+\frac{1}{Y_{a,b-1}}\right)^{-1}, \tag{5.20}$$

as long as $(a, b) \neq (2, \pm 2)$.

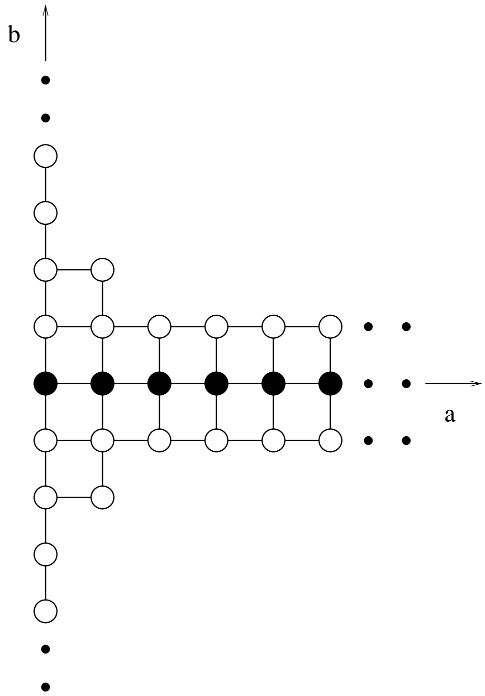

Figure 2. The *AdS/CFT* diagram.

Our Y-diagram shares its structure with that in figure 1 of [37]. Yet, we shall remark the exact parallel to what we have noted at the end of section 4 about the Hubbard model: to close completely the diagram by using the 'standard' rules, we would need two extra equations:

$$Y^{\alpha}_{y^*}(q^+)Y^{\alpha}_{y^*}(q^-) = \left(1 + Y^{\alpha}_{w,1}(\lambda)\right)\left(1 + \frac{1}{Y^{\alpha}_{v,1}(\lambda)}\right)^{-1} \quad (\alpha = 1, 2). \tag{5.21}$$

A careful reader may have noted that we did not prove these equations, since for the nodes $(2, \pm 2)$ we already have the identification

$$Y^{\alpha}_{2,\pm 2}(q) = \frac{1}{Y^{\alpha}_{1,\pm 1}(\pi - q)}, \tag{5.22}$$

and thus, at any rate, we do not need to include the associated equations in the TBA system. A careful analysis suggests that equation (5.21) is in general incorrect.

6. Partial conclusions and remarks

In a nutshell, we have proposed the TBA equations which should control the energy/dimension spectrum of the $AdS_5 \times S^5$ correspondence. We have also derived from them the universal Y-system which should characterize any state of the theory for any value of the coupling constant g. Of course, since universal, this system contains the information about a specific state in a much more involved way.

Nevertheless, we may still rely on the massive integrable field theories. In this area a clear procedure has been established to extract excited-state nonlinear integral equations from that of the ground state; this is initially described from three different perspectives in the papers [29–31]. Essentially, it proves the recipe to extract suitable driving terms $\sum_i \ln S(u_i, u)$ as residues of the convolution integrals, and these terms clearly involve the scattering matrix elements. Under the perspective of the nonlinear integral equation, this idea has been already applied to some sectors of the asymptotic Beisert–Staudacher equations [41–43].

Re-echoing the title of [44], the Hubbard model excursion seems to be still on in this discipline. In fact, we have found just two copies of this model, talking through their massive nodes. Moreover, this is also the structure of the Y-system recently proposed by Gromov, Kazakov and Viera (somehow on symmetry grounds) [37].

Despite the lack of a BA or integrability description of sufficiently *short* operators, we may consider all these arguments in favour of a TBA description of the correspondence.

Note added to the journal version

The (first versions of the) papers [47, 48][9] appeared while we were preparing the present revised version by adding some specifications in the text and other minor corrections in the formulae. Those papers' authors derived independently the TBA equations and the Y-system by making use of the same methods. After considering some evident typos and the different parametrizations adopted in the original manuscripts, our main equations match those of [48]. The relationship between the new equations and the Hubbard model is not discussed in [48], which nevertheless contains, among other results, many interesting comments on the analytic properties of the Y-functions and a derivation of the universal form of the TBA. For what concerns [47] the Y-system is consistent with our findings and those of [48] and, apart from its initial discrepancy in some of the TBA kernels—amended in subsequent versions (cf also the discussion at the end of section 5 in [48])—the TBA equations coincide with ours as well. Moreover, [47] starts the discussion on the generalization to particular families of excited states, along the lines we anticipated in our final section 6.

Finally, the results and final comments of section 5 show the need for careful consideration of our Y-system—directly stemming from the TBA equations—once the lhs is taken as the node $Y_{2,\pm 2}$: strictly speaking the functional relations (5.21) do not hold in these cases. This suggests, at first sight, a significant difference with the Y-system inferred in [37] on symmetry grounds. However, from the discussion in the first paragraph after equation (1) in [37], it is clear that the definition of the Y-system for $Y_{2,\pm 2}$ is ambiguous there, because of the required boundary conditions. Thus, this still supports our results and stresses once more the larger content of information within the TBA integral equations, as already known for relativistic theories [27]. There are also open issues concerning the analytic continuation of the Y-system outside the strip, $-2 < \mathrm{Re}(u) < 2$, that may lead to a more serious disagreement with the proposal of [37]. These have been partially addressed in [48] and need extra work in order to be fully clarified.

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[9] DF thanks G Arutyunov for a copy of the manuscript [48] one day before it should appear.

Fortemente Correlati all'Equilibrio e Fuori Equilibrio: Risultati Esatti e Metodi di Teoria dei Campi'.

Appendix A.

Here we report the definitions used for the kernels involved in the TBA equations (3.5)–(3.8):

$$\phi_{sl(2)}^{Q'Q}(\tilde{p}',\tilde{p}) = \frac{1}{2\pi i}\frac{\partial}{\partial \tilde{p}'}\ln S_{(2)}^{Q'Q}(\tilde{p}',\tilde{p}), \tag{A.1}$$

$$\phi_{xy}^{Q}(\tilde{p},q) = \frac{1}{2\pi i}\frac{\partial}{\partial \tilde{p}}\ln\left(\frac{x^{Q-}(\tilde{p})-y(q)}{x^{Q+}(\tilde{p})-y(q)}\sqrt{\frac{x^{Q+}(\tilde{p})}{x^{Q-}(\tilde{p})}}\right), \tag{A.2}$$

$$\phi_{xv}^{QM}(\tilde{p},\lambda) = \frac{1}{2\pi i}\frac{\partial}{\partial \tilde{p}}\ln S_{xv}^{QM}(\tilde{p},\lambda), \tag{A.3}$$

$$\phi_{vx}^{KQ}(\lambda,\tilde{p}) = -\frac{1}{2\pi i}\frac{\partial}{\partial \lambda}\ln S_{xv}^{QK}(\tilde{p},\lambda), \tag{A.4}$$

$$\phi_{yx}^{Q}(q,\tilde{p}) = \frac{1}{2\pi i}\frac{\partial}{\partial q}\ln\left(\frac{y(q)-x^{Q+}(\tilde{p})}{y(q)-x^{Q-}(\tilde{p})}\sqrt{\frac{x^{Q-}(\tilde{p})}{x^{Q+}(\tilde{p})}}\right), \tag{A.5}$$

$$\phi_{yv}^{K}(q,\lambda) = \phi_{yw}^{K}(q,\lambda) = \frac{1}{2\pi i}\frac{\partial}{\partial q}\ln\left(\frac{v(q)-2\lambda+iK/g}{v(q)-2\lambda-iK/g}\right), \tag{A.6}$$

$$\phi_{vv}^{MK}(\lambda',\lambda) = \phi_{ww}^{MK}(\lambda',\lambda) = \frac{1}{2\pi i}\frac{\partial}{\partial \lambda'}\ln S_{MK}(2\lambda'-2\lambda), \tag{A.7}$$

$$\phi_{vy}^{K}(\lambda,q) = -\phi_{wy}^{K}(\lambda,q) = \frac{1}{2\pi i}\frac{\partial}{\partial \lambda}\ln\left(\frac{2\lambda-v(q)+iK/g}{2\lambda-v(q)-iK/g}\right), \tag{A.8}$$

where we have defined (as in the main text, with the possible omission of the subscript for λ)

$$x^{Q\pm}(\tilde{p}) = \frac{1}{2g}\left(\sqrt{1+\frac{4g^2}{Q^2+\tilde{p}^2}}\mp 1\right)(\tilde{p}-iQ), \tag{A.9}$$

$$\tilde{p}(u) = \frac{ig}{2}\left(\sqrt{4-\left(u+i\frac{Q}{g}\right)^2}-\sqrt{4-\left(u-i\frac{Q}{g}\right)^2}\right), \tag{A.10}$$

$$y(q) = ie^{-iq}, \qquad v(q) = 2\sin(q) = 2\lambda_v, \qquad w(\lambda) = 2\lambda_w, \tag{A.11}$$

$$v_K^{\pm}(\lambda) = 2\lambda_v \pm \frac{iK}{g}, \qquad w_K^{\pm}(\lambda) = 2\lambda_w \pm \frac{iK}{g}, \tag{A.12}$$

$$x(u) = \frac{1}{2}(u - i\sqrt{4-u^2}), \qquad x^{Q\pm}(u) = x\left(u \pm i\frac{Q}{g}\right), \tag{A.13}$$

$$x^{Q\pm}(-u) = -\frac{1}{x^{Q\mp}(u)}, \qquad \tilde{p}(-u) = -\tilde{p}(u). \tag{A.14}$$

It is easy to note that some of these kernels depend only on the difference of the rapidities, as in the relativistic case. They are

$$\phi^{M}_{vy}(\lambda, q) = -\phi^{M}_{wy}(\lambda, q) = \phi_M(\lambda - \sin(q)), \quad \text{where } \phi_M(\lambda) = \frac{1}{2\pi \mathrm{i}} \frac{\partial}{\partial\lambda} \ln\left(\frac{\lambda + \mathrm{i}M/2g}{\lambda - \mathrm{i}M/2g}\right) \tag{A.15}$$

$$\phi^{MK}_{vv}(\lambda', \lambda) = \phi^{MK}_{ww}(\lambda', \lambda) = \phi_{MK}(\lambda' - \lambda), \quad \text{where } \phi_{MK}(\lambda) = \frac{1}{2\pi \mathrm{i}} \frac{\partial}{\partial\lambda} \ln S_{MK}(2\lambda). \tag{A.16}$$

Appendix B.

Here we want to show how also the other kernels satisfy an identity of type (5.2). As far as the kernel

$$\begin{aligned}\phi^{QQ'}_{sl(2)}(u, u') = \frac{1}{2\pi \mathrm{i}} \frac{\partial}{\partial \tilde{p}} \ln \Bigg[&\left(\frac{u - u' + \mathrm{i}\frac{|Q-Q'|}{g}}{u - u' - \mathrm{i}\frac{|Q-Q'|}{g}}\right)\left(\frac{u - u' + \mathrm{i}\frac{Q+Q'}{g}}{u - u' - \mathrm{i}\frac{Q+Q'}{g}}\right) \\ &\times \left[\left(\prod_{j=1}^{Q}\prod_{h=1}^{Q'} \frac{1 - \frac{1}{x(u+\mathrm{i}(Q+2-2j)/g)x(u'+\mathrm{i}(Q'-2h)/g)}}{1 - \frac{1}{x(u+\mathrm{i}(Q-2j)/g)x(u'+\mathrm{i}(Q'+2-2h)/g)}}\right)\sigma_{Q,Q'}(u, u')\right]^{-2} \\ &\times \prod_{k=1}^{\min(Q,Q')-1} \left(\frac{u - u' + \mathrm{i}\frac{|Q-Q'|+2k}{g}}{u - u' - \mathrm{i}\frac{|Q-Q'|+2k}{g}}\right)^2 \Bigg]\end{aligned} \tag{B.1}$$

is concerned, we may shift on the second variable

$$\begin{aligned}&\phi^{QQ'}_{sl(2)}\left(u, u' + \frac{\mathrm{i}}{g}\right) + \phi^{QQ'}_{sl(2)}\left(u, u' - \frac{\mathrm{i}}{g}\right) = \frac{1}{2\pi \mathrm{i}} \frac{\partial}{\partial \tilde{p}} \Bigg[\ln\left(\frac{u - u' + \mathrm{i}\frac{|Q-Q'|}{g} - \frac{\mathrm{i}}{g}}{u - u' - \mathrm{i}\frac{|Q-Q'|}{g} - \frac{\mathrm{i}}{g}}\right) \\ &+ \ln\left(\frac{u - u' + \mathrm{i}\frac{Q+Q'-1}{g}}{u - u' - \mathrm{i}\frac{Q+Q'+1}{g}}\right) + 2\sum_{k=1}^{\min(Q,Q'-1)-1} \ln\left(\frac{u - u' + \mathrm{i}\frac{|Q-Q'|+2k}{g} - \frac{\mathrm{i}}{g}}{u - u' - \mathrm{i}\frac{|Q-Q'|+2k}{g} - \frac{\mathrm{i}}{g}}\right) \\ &+ \ln\left(\frac{u - u' + \mathrm{i}\frac{|Q-Q'|}{g} + \frac{\mathrm{i}}{g}}{u - u' - \mathrm{i}\frac{|Q-Q'|}{g} + \frac{\mathrm{i}}{g}}\right) + \ln\left(\frac{u - u' + \mathrm{i}\frac{Q+Q'+1}{g}}{u - u' - \mathrm{i}\frac{Q+Q'-1}{g}}\right) \\ &+ 2\sum_{k=1}^{\min(Q,Q'+1)-1} \ln\left(\frac{u - u' + \mathrm{i}\frac{|Q-Q'|+2k}{g} + \frac{\mathrm{i}}{g}}{u - u' - \mathrm{i}\frac{|Q-Q'|+2k}{g} + \frac{\mathrm{i}}{g}}\right) - 2\ln\left(\frac{1 - \frac{1}{x\left(u - \frac{\mathrm{i}Q}{g}\right)x\left(u + \mathrm{i}\frac{Q'+1}{g}\right)}}{1 - \frac{1}{x\left(u + \frac{\mathrm{i}Q}{g}\right)x\left(u - \mathrm{i}\frac{Q'-1}{g}\right)}}\right) \\ &- 2\ln\left(\frac{1 - \frac{1}{x\left(u - \frac{\mathrm{i}Q}{g}\right)x\left(u + \mathrm{i}\frac{Q'-1}{g}\right)}}{1 - \frac{1}{x\left(u + \frac{\mathrm{i}Q}{g}\right)x\left(u - \mathrm{i}\frac{Q'+1}{g}\right)}}\right) \\ &- 2\mathrm{i}\sum_{r=2}^{\infty}\sum_{\nu=0}^{\infty} \beta_{r,r+1+2\nu}(g)\big[q^{Q}_{r}(u)q^{Q'+1}_{r+1+2\nu}(u') - q^{Q'+1}_{r}(u')q^{Q}_{r+1+2\nu}(u) \\ &+ q^{Q}_{r}(u)q^{Q'-1}_{r+1+2\nu}(u') - q^{Q'-1}_{r}(u')q^{Q}_{r+1+2\nu}(u)\big]\Bigg],\end{aligned} \tag{B.2}$$

thus proving the identity used in the main text:

$$\phi^{QQ'}_{sl(2)}\left(u,u'+\frac{i}{g}\right)+\phi^{QQ'}_{sl(2)}\left(u,u'-\frac{i}{g}\right)=\sum_{Q''=1}^{\infty}I_{Q'Q''}\phi^{QQ''}_{sl(2)}(u,u')-\delta(u-u')I_{QQ'}. \tag{B.3}$$

Leaving for future consideration the convergence problem of the dressing factor in the mirror theory, we may still think as possible to define somehow (in a convergent or asymptotic sense) the bound state charges q^Q_r as in [45, 46] and then the shifted charges above as

$$q^{Q\pm1}_r(u)=\frac{i}{r-1}\left[\left(\frac{1}{x\left(u+\frac{i(Q\pm1)}{g}\right)}\right)^{r-1}-\left(\frac{1}{x\left(u-\frac{i(Q\mp1)}{g}\right)}\right)^{r-1}\right]. \tag{B.4}$$

Analogously, by direct computation

$$\begin{aligned}\phi^{MQ}_{vx}&\left(\lambda,x^{Q\pm}\left(u+\frac{i}{g}\right)\right)+\phi^{MQ}_{vx}\left(\lambda,x^{Q\pm}\left(u-\frac{i}{g}\right)\right)=-\frac{1}{2\pi i}\frac{\partial}{\partial\lambda}\left[\ln\left(\frac{x^{(Q-1)-}-x\left(v+\frac{iM}{g}\right)}{x^{(Q+1)+}-x\left(v+\frac{iM}{g}\right)}\right)\right.\\&+\ln\left(\frac{x^{(Q+1)-}-x\left(v-\frac{iM}{g}\right)}{x^{(Q-1)+}-x\left(v-\frac{iM}{g}\right)}\right)+\ln\left(\frac{x^{(Q-1)-}-x\left(v-\frac{iM}{g}\right)}{x^{(Q+1)+}-x\left(v-\frac{iM}{g}\right)}\right)+\ln\left(\frac{x^{(Q+1)+}}{x^{(Q-1)-}}\right)\\&+\ln\left(\frac{x^{(Q-1)+}}{x^{(Q+1)-}}\right)+\ln\left(\frac{x^{(Q+1)-}-x\left(v+\frac{iM}{g}\right)}{x^{(Q-1)+}-x\left(v+\frac{iM}{g}\right)}\right)+\sum_{j=1}^{M-1}\left[\ln\left(\frac{u-i\frac{Q-1}{g}-\left(v-\frac{iM}{g}\right)-\frac{2i}{g}j}{u+i\frac{Q+1}{g}-\left(v+\frac{iM}{g}\right)+\frac{2i}{g}j}\right)\right.\\&\left.\left.+\ln\left(\frac{u-i\frac{Q+1}{g}-\left(v-\frac{iM}{g}\right)-\frac{2i}{g}j}{u+i\frac{Q-1}{g}-\left(v+\frac{iM}{g}\right)+\frac{2i}{g}j}\right)\right]\right]\end{aligned} \tag{B.5}$$

($v=2\lambda$), we may prove an identity with the same form, but involving ϕ^{MQ}_{vx},

$$\begin{aligned}\phi^{MQ}_{vx}&\left(\lambda,x^{Q\pm}\left(u+\frac{i}{g}\right)\right)+\phi^{MQ}_{vx}\left(\lambda,x^{Q\pm}\left(u-\frac{i}{g}\right)\right)\\&=\sum_{Q'=1}^{\infty}I_{QQ'}\phi^{MQ'}_{vx}(\lambda,x^{Q'}(u))+\delta(\lambda-u/2)\delta_{Q-1,M}.\end{aligned} \tag{B.6}$$

An identity with the same form may be derived for ϕ^{QM}_{xv}:

$$\begin{aligned}\phi^{QM}_{xv}&\left(x^{Q\pm}(u),\lambda+\frac{i}{2g}\right)+\phi^{QM}_{xv}\left(x^{Q\pm}(u)\lambda-\frac{i}{2g}\right)=\frac{1}{2\pi i}\frac{\partial}{\partial\tilde{p}}\left[\ln\left(\frac{x^{Q-}-x\left(v+\frac{i(M+1)}{g}\right)}{x^{Q+}-x\left(v+\frac{i(M+1)}{g}\right)}\right)\right.\\&+\ln\left(\frac{x^{Q-}-x\left(v-\frac{i(M-1)}{g}\right)}{x^{Q+}-x\left(v-\frac{i(M-1)}{g}\right)}\right)+\ln\left(\frac{x^{Q-}-x\left(v-\frac{i(M+1)}{g}\right)}{x^{Q+}-x\left(v-\frac{i(M+1)}{g}\right)}\right)+2\ln\left(\frac{x^{Q+}}{x^{Q-}}\right)\\&+\ln\left(\frac{x^{Q-}-x\left(v+\frac{i(M-1)}{g}\right)}{x^{Q+}-x\left(v+\frac{i(M-1)}{g}\right)}\right)+\sum_{j=1}^{M-1}\left[\ln\left(\frac{u-i\frac{Q}{g}-\left(v-i\frac{M-1}{g}\right)-\frac{2i}{g}j}{u+i\frac{Q}{g}-\left(v+i\frac{M+1}{g}\right)+\frac{2i}{g}j}\right)\right.\\&\left.\left.+\ln\left(\frac{u-i\frac{Q}{g}-\left(v-i\frac{M+1}{g}\right)-\frac{2i}{g}j}{u+i\frac{Q}{g}-\left(v+i\frac{M-1}{g}\right)+\frac{2i}{g}j}\right)\right]\right],\end{aligned} \tag{B.7}$$

$$\begin{aligned}\phi^{QM}_{xv}&\left(x^{Q\pm}(u),\lambda+\frac{i}{2g}\right)+\phi^{QM}_{xv}\left(x^{Q\pm}(u),\lambda-\frac{i}{2g}\right)\\&=\sum_{M'=1}^{\infty}I_{MM'}\phi^{QM'}_{xv}(x^{Q}(u),\lambda)+\delta(\lambda-u/2)\delta_{Q-1,M}.\end{aligned} \tag{B.8}$$

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